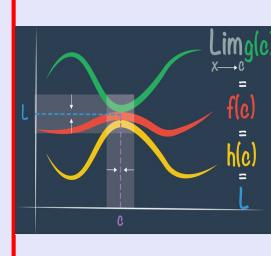


# Math 261

## Spring 2023

### Lecture 36



Feb 19-8:47 AM

Verify the conditions by Rolle's theorem for  
 $f(x) = \sqrt{x} - \frac{1}{3}x$  on  $[0, 9]$ , and find all  
numbers that satisfy the conclusion of  
Rolle's theorem.

$$f(x) = \sqrt{x} - \frac{1}{3}x \text{ is cont. on } [0, 9]$$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3} \quad f(x) \text{ is diff. on } (0, 9)$$

$$f(0) = 0, \quad f(9) = 0 \Rightarrow f(0) = f(9)$$

Conclusion  $f'(c) = 0$  for  $(0, 9)$

$$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0 \quad \frac{1}{2\sqrt{c}} = \frac{1}{3} \quad 2\sqrt{c} = 3$$

$$c = \left(\frac{3}{2}\right)^2$$

$$c = \frac{9}{4}$$

$$c = 2.25$$

Apr 19-8:45 AM

Verify the conditions of MVT for  
 $f(x) = \sqrt[3]{x}$  on  $[0, 1]$ , then find all numbers  $c$  that satisfy the conclusion of MVT.

$$f(x) = \sqrt[3]{x} \quad f(x) \text{ is cont. on } [0, 1]$$

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}} \quad f(x) \text{ is diff. on } (0, 1)$$

$$f(0) = 0, \quad f(1) = 1 \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{3\sqrt[3]{c^2}} = \frac{1-0}{1-0} \quad \frac{1}{3\sqrt[3]{c^2}} = 1 \quad 3\sqrt[3]{c^2} = 1$$

$$\sqrt[3]{c^2} = \frac{1}{3}$$

$$c^2 = \frac{1}{27} \\ c = \sqrt{\frac{1}{27}}$$

$$c = \frac{1}{3\sqrt{3}} \\ c = \frac{\sqrt{3}}{9}$$

$(0, 1)$

$$c \approx 0.192$$

Apr 19-8:51 AM

Suppose  $3 \leq f'(x) \leq 5$  for all values of  $x$ .

Show that  $18 \leq f(8) - f(2) \leq 30$ .

Since  $3 \leq f'(x) \leq 5$  for all values of  $x$

therefore  $f(x)$  is cont. for all values of  $x$ .

Consider  $[2, 8]$ , by MVT

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(8) - f(2)}{8 - 2} = \frac{f(8) - f(2)}{6}$$

Since  $3 \leq f'(x) \leq 5$ , then

$$3 \leq \frac{f(8) - f(2)}{6} \leq 5$$

Multiply by 6

$$6 \cdot 3 \leq f(8) - f(2) \leq 6 \cdot 5$$

$$18 \leq f(8) - f(2) \leq 30$$

Apr 19-8:57 AM

Is there a function  $f(x)$  such that

$$f(0) = -1, \quad f(2) = 4, \quad \text{and} \quad f'(x) \leq 2 \quad \text{for all } x?$$

Consider  $[0, 2]$  and apply MVT,

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - (-1)}{2} = \frac{5}{2}$$

$$f'(c) = \frac{5}{2} \quad \text{however} \quad f'(x) \leq 2 \quad \frac{5}{2} \leq 2 \quad \text{false}$$

no such function exists.

Apr 19-9:03 AM

Find the area of the largest rectangle that can be inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Centered at  $(0,0)$

$x$ -Ints  $(\pm a, 0)$

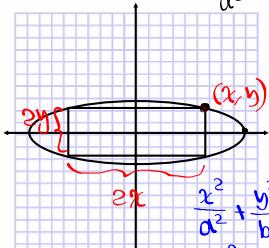
$y$ -Int  $(0, \pm b)$

Symmetric

1)  $x$ -axis

2)  $y$ -axis

3) Origin



$$\text{Area} = 4xy$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\text{Area} = 4xy$$

$$= 4x \cdot \frac{b}{a} \sqrt{a^2 - x^2}$$

$$f(x) = \frac{4b}{a} x \sqrt{a^2 - x^2}$$

$$f'(x) =$$

$$f''(x) =$$

$$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right)$$

$$= b^2 \left(\frac{a^2}{a^2} - \frac{x^2}{a^2}\right)$$

$$= \frac{b^2}{a^2} (a^2 - x^2)$$

$$y = \frac{b}{a} \sqrt{a^2 - x^2}$$

$$f'(x) > 0$$

$$f''(x) < 0 \quad \text{C.D.} \quad \text{Max}$$

Apr 18-9:34 AM

$$\begin{aligned}
 f(x) &= \frac{4b}{a} x \sqrt{a^2 - x^2} \quad f(x) = \frac{4b}{a} \cdot x \cdot (a^2 - x^2)^{\frac{1}{2}} \\
 f'(x) &= \frac{4b}{a} \left[ 1 \cdot (a^2 - x^2)^{\frac{1}{2}} + x \cdot \frac{1}{2} (a^2 - x^2)^{-\frac{1}{2}} \cdot -2x \right] \\
 &= \frac{4b}{a} \left[ \sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} \right] \\
 &= \frac{4b}{a} \left[ \frac{\sqrt{a^2 - x^2} \cdot \sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}} - \frac{x^2}{\sqrt{a^2 - x^2}} \right] \\
 &= \frac{4b}{a} \left[ \frac{a^2 - x^2 - x^2}{\sqrt{a^2 - x^2}} \right] = \frac{4b}{a} \cdot \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}
 \end{aligned}$$

$f'(x) = 0 \rightarrow a^2 - 2x^2 = 0 \rightarrow x = \frac{a}{\sqrt{2}}$   
 $f(x)$  undefined  $\rightarrow a^2 - x^2 = 0 \rightarrow x = a \times$   
 $f'(x) = \frac{4b}{a} \cdot \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$

$x=0$	$\frac{a}{\sqrt{2}}$	$x=a$
$f' > 0$		$f' < 0$

max happens at  $x = \frac{a}{\sqrt{2}}$

First Derivative Test

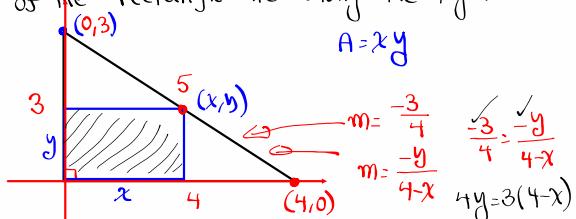
$y = \frac{b}{a} \sqrt{a^2 - x^2} = \frac{b}{a} \sqrt{a^2 - \frac{a^2}{2}} = \frac{b}{\sqrt{2}}$

max. area  $4xy = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = [2ab]$

$f(x)$  is max at  $\frac{a}{\sqrt{2}}$

Apr 19-9:10 AM

Find the area of the **largest rectangle** that can be inscribed in a right triangle with legs of 3cm and 4cm if two sides of the rectangle lie along the legs.



$$\text{Area} = xy \qquad y = \frac{3(4-x)}{4}$$

$$f(x) = x \cdot \frac{3(4-x)}{4} = \frac{3}{4}[4x - x^2]$$

$$f'(x) = \frac{3}{4}[4 - 2x] = \frac{3}{4} \cdot 2(2-x) = \frac{3}{2}(2-x)$$

$$f''(x) = \frac{3}{2}(-1) = \frac{3}{2} < 0 \quad \text{CD}$$

$$x=2$$

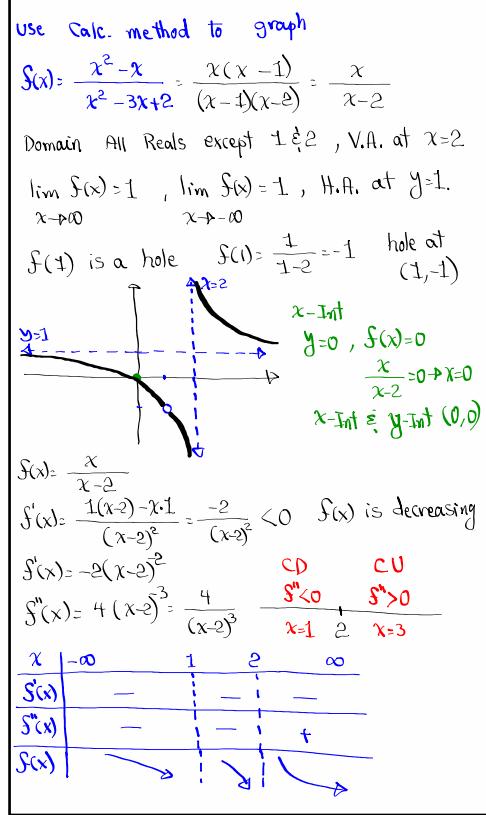
$$y = \frac{3}{4}(4-2) = \frac{3}{2}$$

$$\text{Area} = xy$$

$$= 2 \cdot \frac{3}{2} = [3 \text{ m}^2]$$



Apr 18-9:46 AM

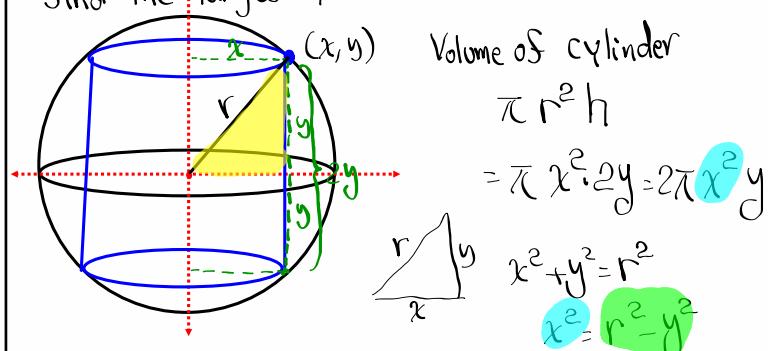


Apr 19-9:30 AM

Consider a sphere with radius  $r$ ,

inscribe a circular cylinder in the Sphere.

Find the largest possible volume of such cylinder.



$$V(y) = 2\pi(r^2 - y^2) \cdot y$$

$$V'(y)$$

Discuss concavity

$$V''(y)$$

Discuss max or min.

Apr 19-9:40 AM